

On Integrable Models and their Interrelations

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Abstract

We present an elementary discussion of the Calogero-Moser model. This gives us an opportunity to illustrate basic concepts of the dynamical integrable models. Some ideas are also presented regarding interconnections between integrable models based on the relation established between the Calogero-Moser model and the truncated KP hierarchy of Burgers-Hopf type.

1. Introduction. Calogero-Moser Model

The main purpose of this talk is to describe, in an elementary way, the notion of integrability, its Lax formulation and the relations, which can be established between the various integrable models despite their different appearances and origins.

The subject of integrability is currently of widespread interest in view of the recent developments in high energy physics, which brought integrable hierarchies, including the Kadomtsev-Petviashvili one (*KP*), into the central position in the studies of the matrix models known to describe, at multicritical points, $c \leq 1$ matter systems coupled to $D = 2$ quantum gravity [1]. It is furthermore of interest to study interrelations between integrable models as they can reveal new classes of solutions.

We first recall the notion of integrability.

Complete integrability: Consider a Hamiltonian system with coordinates $(\vec{q}, \vec{p}) = (q_1, \dots, q_N, p_1, \dots, p_N)$ possessing a standard Hamiltonian structure with a Hamiltonian $H(p, q)$ and Poisson bracket $\{\cdot, \cdot\}$. We call a system *integrable* if we can write the general solution to the equations of motion in terms of (finitely many) algebraic manipulations, which can include evaluation of integrals in terms of known functions.

A condition for integrability is existence of sufficiently many, independent, Poisson commuting (i.e. in involutions) functions $h_r(p, q)$ ($r = 1, \dots, N$) of \vec{q} and \vec{p} , which are integrals of motion. The fact that the functions h_r, h_s are in involution:

$$\{h_r, h_s\} = \sum_{j=1}^N \left(\frac{\partial h_r}{\partial q_j} \frac{\partial h_s}{\partial p_j} - \frac{\partial h_r}{\partial p_j} \frac{\partial h_s}{\partial q_j} \right) = 0 \quad (1)$$

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implies that the function h_r is constant along the solutions of the Hamiltonian system generated by h_s :

$$\dot{q}_j = \frac{\partial h_s}{\partial p_j} \quad ; \quad \dot{p}_j = -\frac{\partial h_s}{\partial q_j} \quad (2)$$

as follows by inserting (2) into (1)

$$\sum_{j=1}^N \left(\frac{\partial h_r}{\partial q_j} \dot{q}_j + \frac{\partial h_r}{\partial p_j} \dot{p}_j \right) = \dot{h}_r = 0 \quad (3)$$

We will now illustrate the concept of integrability on the specific example of Calogero-Moser model [2, 3] and show how the Lax formulation arises naturally in this context. We first recall the formulation of the problem. Let $X(t)$ be a $N \times N$ Hermitian matrix. We define a model by choosing a most simple dynamics for the matrix $X(t)$. Namely, we let the second time derivative of X be zero:

$$\ddot{X}(t) = 0 \quad \rightarrow \quad X(t) = X(0) + \dot{X}(0)t \quad (4)$$

The dynamics will be described in terms of the eigenvalues of the X -matrix. The next step involves, therefore, a diagonalization of $X(t)$ by a unitary matrix U .

$$X \quad \rightarrow \quad Q(t) = U^{-1}(t)X(t)U(t) = \begin{pmatrix} q_1(t) & & \\ & \ddots & \\ & & q_N(t) \end{pmatrix} \quad (5)$$

where we denoted by $\{q_j \mid j = 1, \dots, N\}$ the eigenvalues of the X matrix. Making use of the identity $\partial U^{-1}(t)/\partial t = -U^{-1}(t)(\partial U(t)/\partial t)U^{-1}(t)$ we find the flows of X :

$$\frac{\partial X}{\partial t} = [(\partial U(t)/\partial t)U^{-1}(t), U(t)QU^{-1}] + U(t)\frac{\partial Q}{\partial t}U^{-1}(t) = U(t)L(t)U^{-1}(t) \quad (6)$$

where we have introduced the matrix L , which is a prototype of a Lax operator:

$$L \equiv [U^{-1}(t)(\partial U(t)/\partial t), Q(t)] + \frac{\partial Q}{\partial t} = [M(t), Q(t)] + \frac{\partial Q}{\partial t} \quad (7)$$

with $M(t) \equiv U^{-1}(t)(\partial U(t)/\partial t)$. Differentiating (6) one more time we obtain:

$$0 = \ddot{X} = U \left([U^{-1}(t)(\partial U(t)/\partial t), L(t)] + \dot{L} \right) U^{-1} \quad (8)$$

which implies the Lax equation of motion:

$$\dot{L} = [L(t), M(t)] \quad (9)$$

The fact that we can cast the flows of dynamical variables of the integrable model in the form of Lax equation signals integrability. In fact, it can be shown that any completely integrable Hamiltonian system admits a Lax representation (at least locally) [4]. The Lax formulation leads straightforwardly to the construction of the integrals of motion. Namely, for any invariant function I , like $I(A) = \text{Tr}(A^k)$ for some matrix A , $I(L)$ is a constant of motion.

For simplicity, we assume that $U(t=0) = 1$, which defines as initial conditions: $Q(0) = X(0)$ and $L(0) = \dot{X}(0)$. Since $\ddot{X} = 0$ the matrix $C = [X, \dot{X}]$ is a constant and therefore given by the initial conditions:

$$C_{ij} = ([Q(0), L(0)])_{ij} \quad \rightarrow \quad L(0)_{ij} = \frac{C_{ij}}{q_i(0) - q_j(0)} \quad i \neq j \quad (10)$$

From (7) we find:

$$L(t)_{ij} = \delta_{ij}\dot{q}_j - M_{ij}(t)(q_i(t) - q_j(t)) \quad \rightarrow \quad M_{ij}(0) = -\frac{C_{ij}}{(q_i(0) - q_j(0))^2} \quad i \neq j \quad (11)$$

Extending this straightforwardly to arbitrary time t we find

$$M_{ij}(t) = -\frac{C_{ij}}{(q_i(t) - q_j(t))^2} \quad ; \quad i \neq j \quad (12)$$

which provides an ansatz consistent with the off-diagonal part of (9), when verified together with $C_{ij} = ig(1 - \delta_{ij})$ and $M_{ii} = ig \sum_{k \neq j} (q_j(t) - q_k(t))^{-2}$ (g is a coupling constant). With these assumptions we construct the Lax pair

$$L_{ij}(t) = \delta_{ij} \dot{q}_j + \frac{ig(1 - \delta_{ij})}{(q_i(t) - q_j(t))} \quad (13)$$

$$M_{ij}(t) = ig \delta_{ij} \sum_{k \neq i} (q_i(t) - q_k(t))^{-2} - \frac{ig(1 - \delta_{ij})}{(q_i(t) - q_j(t))^2} \quad (14)$$

which, when plugged in (9), produces equations of motion. These equations of motion can alternatively be produced by inserting the Hamiltonian:

$$\mathcal{H} = \frac{1}{2} \left(\sum_{i=1}^N p_i^2 + g^2 \sum_{i \neq j} \frac{1}{(q_i(t) - q_j(t))^2} \right) = \frac{1}{2} \text{Tr}(L^2) \quad (15)$$

into the Hamilton equations of motion:

$$\dot{p}_i = \{ \mathcal{H}, p_i \} = 2 \sum_{i \neq j} \frac{g^2}{(q_i(t) - q_j(t))^3} \quad ; \quad \dot{q}_i = \{ \mathcal{H}, q_i \} = p_i \quad (16)$$

The system is completely integrable in a sense that the above equations of motion can be solved by eigenvalues of $X(t) = Q(0) + L(0)t$ with $Q_{ij}(0) = \delta_{ij} q_j(0)$ and $L_{ij}(t=0)$ given by (13).

Having established the Lax representation for the equations of motion it is easy to find the integrals of motion following the standard recipe:

$$H_k = \text{Tr}(L^k) \quad \rightarrow \quad \frac{\partial H_k}{\partial t} = k \text{Tr}([L, M] L^{k-1}) = 0 \quad (17)$$

The presence of higher Hamiltonians signals that there exist “higher” times t_k . The corresponding higher flows are also governed by the Lax equations:

$$\frac{\partial L}{\partial t_k} = [L(t), M_k(t)] \quad (18)$$

with

$$(M_k)_{ij} = \delta_{ij} \sum_{k \neq i} \frac{k}{q_i(t) - q_k(t)} (L^{k-1})_{ik} - (1 - \delta_{ij}) \frac{k}{q_i(t) - q_j(t)} (L^{k-1})_{ij} \quad (19)$$

The Lax equations (18) allow now a direct proof that H_k 's are in involution:

$$\{ H_k, H_l \} = \{ H_k, \text{Tr}(L^l) \} = \frac{\partial \text{Tr}(L^l)}{\partial t_k} = \text{Tr}([L^l, M_k]) = 0 \quad (20)$$

To investigate further the algebraic structure of the Calogero-Moser model we define in addition to H_k also $\mathcal{L}_k = \text{Tr}(QL^{k+1})$ for $k = -1, 0, \dots$, where Q is the diagonal matrix defined in (5). It can be shown [5] that H_k and \mathcal{L}_k enter the Poisson algebra identical to that of the algebra being a semi-direct product of an Abelian Kac-Moody current algebra with the (centerless) Virasoro algebra:

$$\{ H_n, \mathcal{L}_m \} = n H_{n+m} \quad ; \quad \{ \mathcal{L}_n, \mathcal{L}_m \} = (n - m) \mathcal{L}_{n+m} \quad (21)$$

These results follows from the form of flows for L (18) and for Q [6, 5]:

$$\frac{\partial Q}{\partial t_n} = [Q, M_n] + n L^{n-1} \quad (22)$$

Similar structure has also been found in the hierarchy of nonlinear partial differential equations which serves as our second main illustration.

2. The KP hierarchy

An important example of an integrable system admitting the Lax formulation is given by the *KP* hierarchy consisting of the pseudo-differential Lax operator L :

$$L = D + \sum_{i=1}^{\infty} u_i D^{-i} . \quad (23)$$

which enters the following family of Lax equations:

$$\partial_n L = \frac{\partial L}{\partial t_n} = [B_n, L] \quad n = 1, 2, \dots \quad (24)$$

describing isospectral deformations of L . In (24) $t = \{t_n\}$ are the evolution parameters (infinitely many time coordinates) and $B_n = L_+^n$ is the differential part of $L^n = L_+^n + L_-^n = \sum_{i=0}^{\infty} P_i(n) D^i + \sum_{i=-\infty}^{-1} P_i(n) D^i$.

One can also represent the Lax operator in terms of the dressing operator $W = 1 + \sum_{i=1}^{\infty} w_i D^{-i}$ through $L = W D W^{-1}$. In this framework the equation (24) is equivalent to the so called Wilson-Sato equation:

$$\partial_n W \equiv \frac{\partial W}{\partial t_n} = B_n W - W D^n = -L_-^n W \quad (25)$$

Define next the Baker-Akhiezer (BA) function via

$$\psi(t, \lambda) = W e^\xi = w(t, \lambda) e^\xi; \quad w(t, \lambda) = 1 + \sum_1^{\infty} w_n(t) \lambda^{-n} , \quad (26)$$

where

$$\xi(t, \lambda) \equiv \sum_{n=1}^{\infty} t_n \lambda^n \quad ; \quad t_1 = x \quad (27)$$

There is also an adjoint wave function $\psi^* = W^{*-1} \exp(-\xi(t, \lambda)) = w^*(t, \lambda) \exp(-\xi(t, \lambda))$, $w^*(t, \lambda) = 1 + \sum_1^{\infty} w_i^*(t) \lambda^{-i}$, and one has the following linear systems:

$$L\psi = \lambda\psi; \quad \partial_n \psi = B_n \psi; \quad L^* \psi^* = \lambda \psi^*; \quad \partial_n \psi^* = -B_n^* \psi^* . \quad (28)$$

Note that eq.(24) for the *KP* hierarchy flows follows then from the compatibility conditions among these equations.

Also, the *KP* hierarchy has an ‘‘additional’’ symmetry structure [7, 8, 9] similar to the one encountered in the Calogero-Moser model. Define namely

$$Q \equiv W \left(\sum k t_k D^{k-1} \right) W^{-1} \quad (29)$$

We can now supplement (28) by

$$Q\psi = \frac{\partial \psi}{\partial \lambda} \quad \rightarrow \quad [L, Q] = 1 \quad (30)$$

Q enters the evolution equations

$$\frac{\partial Q}{\partial t_n} = [B_n, Q] = [Q, (L^n)_-] + n L^{n-1} \quad (31)$$

which have the same form as (22) in the setting of the Calogero-Moser model.

There exists a quite natural way of describing the *KP* hierarchy based on one single function – the so-called tau function $\tau(t)$. This approach is an alternative to using the Lax operator and the calculus of pseudo-differential operators. The τ -function is related to the BA functions via

$$\psi(t, \lambda) = e^{\xi(t, \lambda)} \frac{\tau(t_i - 1/i\lambda^i)}{\tau(t_i)} = e^{\xi(t, \lambda)} \sum_{n=0}^{\infty} \frac{p_n \left(-\tilde{\partial} \right) \tau(t)}{\tau(t)} \lambda^{-n} \quad (32)$$

$$\psi^*(t, \lambda) = e^{\xi(t, \lambda)} \frac{\tau(t_i + 1/i\lambda^i)}{\tau(t_i)} = e^{\xi(t, \lambda)} \sum_{n=0}^{\infty} \frac{p_n \left(\tilde{\partial} \right) \tau(t)}{\tau(t)} \lambda^{-n} \quad (33)$$

where $\tilde{\partial} = (\partial_1, (1/2)\partial_2, (1/3)\partial_3, \dots)$ and the Schur polynomials p_n are defined through $e^{\xi(t,\lambda)} = \sum_{n=0}^{\infty} p_n(t_1, t_2, \dots, t_j) \lambda^j$. The BA functions enter the fundamental bilinear identity

$$\oint_{\infty} \psi(t, \lambda) \psi^*(t', \lambda) d\lambda = 0 \quad (34)$$

which generates the entire KP hierarchy. In (34) $\oint_{\infty} (\cdot) d\lambda$ is the residue integral about ∞ . It is possible to rewrite the above identity in terms of the tau functions obtaining

$$\oint_{\infty} \tau(t - [\lambda^{-1}]) \tau(t' + [\lambda^{-1}]) e^{\xi(t,\lambda) - \xi(t',\lambda)} d\lambda = 0 \quad (35)$$

Taylor expanding (35) in y ($t \rightarrow t - y$, $t' \rightarrow t + y$) leads to

$$\left(\sum_0^{\infty} p_n(-2y) p_{n+1}(\tilde{D}) e^{\sum_1^{\infty} y_i D_i} \right) \tau \cdot \tau = 0 \quad (36)$$

where D_i is the Hirota derivative defined as

$$D_j^m a \cdot b = (\partial^m / \partial s_j^m) a(t_j + s_j) b(t_j - s_j) |_{s=0} \quad , \quad \tilde{D} = (D_1, (1/2)D_2, (1/3)D_3, \dots) \quad (37)$$

The coefficients of the y_n -expansion in (36) yield

$$\left(\frac{1}{2} D_1 D_n - p_{n+1}(\tilde{D}) \right) \tau \cdot \tau = 0 \quad (38)$$

which are called the Hirota bilinear equations.

3. Truncated KP Hierarchy, Burgers-Hopf hierarchy

Here we consider a class of truncated KP hierarchies constructed from m -truncated dressing operator W :

$$W = \sum_{i=0}^m w_i(t) D^{-i} \quad ; \quad w_i = \frac{p_i(-\tilde{\partial}) \tau(t)}{\tau(t)} \quad (39)$$

The Lax operator is given by the usual relation $L = WDW^{-1}$ and the m -truncated dressing operator W satisfies the Sato equations as in (25).

The generalized Hopf-Cole transformation applied to this problem leads to the differential equation:

$$W \partial^m \phi_k = 0 \quad k = 1, \dots, m \quad (40)$$

It is well-known that, while solutions of the general KP hierarchy form the universal Grassmann manifold UGM, the solutions of (40) defining the m -truncated KP hierarchy form the Grassmann manifold $GM(m, \infty) = \text{Mat}(\infty \times m) / GL(m; \mathbb{C})$ where $\text{Mat}(\infty \times m)$ denotes $\infty \times m$ matrices of rank m [10, 11].

In terms of the solutions of the m -th order differential equation (40) the Wilson-Sato equations take a simpler form:

$$\partial_n \phi_i = \partial^n \phi_i \quad i = 1, \dots, m \quad (41)$$

In different words we have the following lemma:

Lemma. *Eq.(41) is equivalent to the Wilson-Sato equation (25) for the m -truncated dressing operator from (39).*

Let us return to (40). For $W = 1 + w_1 D^{-1} + \dots + w_m D^{-m}$ equation (40) factorizes as follows

$$WD^m \phi_m = (D^m + w_1 D^{m-1} + \dots + w_m) \phi_k = (D + v_m)(D + v_{m-1}) \dots (D + v_1) \phi_k = 0 \quad (42)$$

There is a relation between the coefficients v_i of (42) and the solutions ϕ_k :

$$v_i = \partial \left(\ln \frac{W_{i-1}[\phi_1, \dots, \phi_{i-1}]}{W_i[\phi_1, \dots, \phi_i]} \right) \quad W_0 = 1 \quad (43)$$

For $m = 1$ this relation takes the form of the classical Hopf-Cole transformation:

$$(1 + w_1 D^{-1}) D\phi = 0 \quad \rightarrow \quad (\partial + w_1) \phi = 0 \quad \rightarrow \quad w_1 = -\partial_x \ln \phi \quad (44)$$

The corresponding differential operator takes the form $WD^m = D - \partial_x (\ln \phi) = \phi D \phi^{-1}$. Since for all m we have $WDW^{-1} = (WD^m)D(WD^m)^{-1}$ we are lead to the Lax operator:

$$L^{(1)} = (\phi D \phi^{-1}) D (\phi D^{-1} \phi^{-1}) = D + [\phi (\ln \phi)'''] D^{-1} (\phi)^{-1} \quad (45)$$

One finds from the above Lemma that the Lax equations (24) for $L^{(1)}$ are equivalent to $\partial_n \phi = \partial_x^n \phi$ or in terms of the coefficient $w \equiv -w_1$ of the dressing operator ($W = 1 + w_1 D^{-1}$):

$$\partial_n w = \partial_x P_n(w) \quad (46)$$

$$P_{n+1}(w) = (\partial + w) P_n(w) \quad n = 0, 1, 2, \dots \quad P_0(w) = 1 \quad (47)$$

Here $P_n(w)$ are Faá di Bruno polynomials fully determined by the recurrence relation in (47). The system of nonlinear differential equations in (46) is called Burgers-Hopf hierarchy.

We now relate a class of solution of the Burgers-Hopf hierarchy to the Calogero-Moser model [12]. Let

$$\phi = \prod_{i \in I} (x - q_i(t')) \quad (48)$$

be a solution of (41) with $x = t_1$ and $t' = (t_2, t_3, \dots)$. Note that this is an ansatz for the τ -function of the Burgers-Hopf hierarchy. We find that equation (41) is equivalent to following evolution equation for q_i :

$$\partial_n q_i = -n! \sum_{\substack{\{j_1, \dots, j_{n-1}\} \not\ni i \\ \{j_1 < \dots < j_{n-1}\}}} (q_i - q_{j_1})^{-1} \dots (q_i - q_{j_{n-1}})^{-1} \quad ; \quad n \geq 2 \quad (49)$$

For $n = 2$ we get:

$$\dot{q}_i = \partial_2 q_i = -2 \sum_{j \neq i} \frac{1}{(q_i - q_j)} \quad (50)$$

which leads to

$$\ddot{q}_i = 8 \sum_{j \neq i} \frac{1}{(q_i - q_j)^3} \quad (51)$$

Hence, the solution of Burgers-Hopf hierarchy of the type shown in (48) can be embedded in the Hamiltonian system (16). Equation (50), when compared with (51), suggests to obtain the system from (48) via hamiltonian reduction of Calogero-Moser model by imposing constraints $\varphi_i = \phi_i - 2 \sum_{j \neq i} 1/(q_i - q_j) = 0$. These constraints turn out to be first class and consequently the class of solutions (48) of the Burgers-Hopf model can not be obtained from the Calogero-Moser model by Hamiltonian reduction. In fact, imposing $\varphi_i = 0$ puts all Hamiltonians H_k of the Calogero-Moser model to zero.

If, however, (48) is extended to be an ansatz for the τ -function of the complete KP hierarchy (with untruncated dressing operator) then, as shown in [13], the flows of q_i 's obey the equations of motion (16) with Calogero-Moser Hamiltonians given in (17).

We generalize now the Burgers-Hopf hierarchy by applying successively the Darboux-Bäcklund transformations to the Burgers-Hopf Lax structure (45). This leads to a special realization of the truncated KP hierarchy in terms of one function ϕ only. The resulting hierarchy we call generalized Burgers-Hopf hierarchy. The Lax structure obtained in this process takes the following form of recursive relations:

$$L^{(k+1)} = \left(\Phi^{(k)} D \Phi^{(k)-1} \right) L^{(k)} \left(\Phi^{(k)} D^{-1} \Phi^{(k)-1} \right) = D + \Phi^{(k+1)} D^{-1} \Psi^{(k+1)} \quad (52)$$

$$\Phi^{(k+1)} = \Phi^{(k)} \left(\ln \Phi^{(k)} \right)'' + \left(\Phi^{(k)} \right)^2 \Psi^{(k)} \quad , \quad \Psi^{(k+1)} = \left(\Phi^{(k)} \right)^{-1} \quad (53)$$

with $\Phi^{(0)} = \phi$.

It is known [14] that the Darboux-Bäcklund transformations induce semi-infinite Toda chain structure on the sequence of the Burgers-Hopf hierarchies. Consequently the τ -function of the semi-infinite Toda chain belongs to the Darboux-Bäcklund group orbit of the trivial τ -function (vacuum) of the generalized Burgers-Hopf system.

These remarks obtain a special relevance in view of the relation of the generalized Burgers-Hopf hierarchy to the one-matrix string model. We shall consider one-matrix model with partition function (M is a Hermitian $N \times N$ matrix) :

$$\begin{aligned} Z_N[t] &= \int dM \exp \left\{ \sum_{r=1}^{\infty} t_r \text{Tr} M^r \right\} \\ &= \frac{1}{N!} \int \prod_{i=1}^N d\lambda_i \exp \left\{ \left(\sum_{i=1}^N \sum_{k=1}^{\infty} t_k \lambda^k \right) \right\} \prod_{i>j=1}^N (\lambda_i - \lambda_j)^2 \end{aligned} \quad (54)$$

which gives rise to a Toda matrix hierarchy system of evolution equations:

$$\frac{\partial}{\partial t_r} Q = [Q_{(+)}^r, Q] \quad (55)$$

Choosing parametrization $Q_{nn} = a_0(n)$, $Q_{n,n-1} = a_1(n)$ (the rest being zero) for the matrix Q , we can cast the matrix hierarchy equations (55) in the form of a discrete linear system

$$\partial \Psi_n = \Psi_{n+1} + a_0(n) \Psi_n \quad (56)$$

$$\begin{aligned} \lambda \Psi_n &= \Psi_{n+1} + a_0(n) \Psi_n + a_1(n) \Psi_{n-1} \\ &= L_n \Psi_n \quad n \geq 0 \end{aligned} \quad (57)$$

where the Lax operator L_n associated to the site n can be written as

$$L_n = \partial + a_1(n) \frac{1}{\partial - a_0(n-1)} \quad (58)$$

and with compatibility conditions for (56)–(57) having the form of the Toda lattice equations of motion:

$$\partial a_0(n) = a_1(n+1) - a_1(n) \quad (59)$$

$$\partial a_1(n) = a_1(n) (a_0(n) - a_0(n-1)) \quad (60)$$

In fact the lattice jump $n \rightarrow n+1$ can be given a meaning of Darboux-Bäcklund transformation [14] within the generalized Burgers-Hopf hierarchy. To see this we rewrite (56) as follows:

$$\Psi_{n+1} = e^{\int a_0(n)} \partial e^{-\int a_0(n)} \Psi_n \quad (61)$$

or in a equivalent form obtained taking into account (60)

$$\Psi_{n+1} = a_1(n) e^{\int a_0(n-1)} \partial \left(a_1(n) e^{\int a_0(n-1)} \right)^{-1} \Psi_n = \Phi(n) \partial \Phi^{-1}(n) \Psi_n \equiv T(n) \Psi_n \quad (62)$$

where $\Phi(n) = a_1(n) e^{\int a_0(n-1)}$ and $T(n) = \Phi(n) \partial \Phi^{-1}(n)$ plays a role of the Darboux-Bäcklund transformation operator generating the lattice translation $n \rightarrow n+1$. We find

$$\begin{aligned} L_{n+1} \Psi_{n+1} &= \lambda (\partial - a_0(n)) \Psi_n \\ &= (\partial - a_0(n)) L_n \Psi_n \\ &= (\partial - a_0(n)) L_n (\partial - a_0(n))^{-1} \Psi_{n+1} \end{aligned} \quad (63)$$

So, the Lax operators at different sites are related by a Darboux-Bäcklund transformation:

$$L_{n+1} = (\partial - a_0(n)) L_n (\partial - a_0(n))^{-1} = T(n) L_n T^{-1}(n) \quad (64)$$

where L_n itself takes the form:

$$L_n = \partial + \Phi(n) \partial^{-1} \Psi(n) \quad , \quad \Phi(n) = a_1(n) e^{\int a_0(n-1)} \quad , \quad \Psi(n) \equiv e^{-\int a_0(n-1)} \quad (65)$$

Here L_n has a form of the operator belonging to the generalized Burgers-Hopf hierarchy. Recalling that $a_1(n=0) = 0$ and therefore $L_0 = \partial$ we see that we proved:

Proposition. *The one-matrix model problem is equivalent to the generalized Burgers-Hopf system possessing the symmetry with respect to the Darboux-Bäcklund transformations.*

This identifies the partition function of the discrete one-matrix model as a τ function of the generalized Burgers-Hopf system having Wronskian form [14].

This talk focused on the linkage between various integrable systems. From the simple realization of integrability in the Calogero-Moser model to the complex structure of the KP hierarchy of the non-linear differential equations we have emphasized the common features involving the Lax pair formulation and the τ -function construction. The relations between different models like discrete matrix models, Toda hierarchy and continuum hierarchies of non-linear differential equations are of current interest in theoretical physics and are among the important tools used to find the τ -function solutions of integrable models.

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eq:invo	1	Page:	1
eq:hameq	2	Page:	2
eq:consth	3	Page:	2
eq:ddot	4	Page:	2
eq:uxu	5	Page:	2
eq:xovert	6	Page:	2
eq:lmat	7	Page:	2
eq:xddot	8	Page:	2
eq:laxeq	9	Page:	2
eq:lzeroij	10	Page:	2
eq:mijzero	11	Page:	2
eq:mijt	12	Page:	2
eq:lijt	13	Page:	3
eq:mijta	14	Page:	3
eq:hamfct	15	Page:	3
eq:hameqs	16	Page:	3
eq:recipe	17	Page:	3
eq:laxeqe	18	Page:	3
eq:mijte	19	Page:	3
eq:recip	20	Page:	3
eq:kmvalg	21	Page:	3
eq:qeqe	22	Page:	3
eq:kpl	23	Page:	4
eq:lax-eq	24	Page:	4
eq:sato-a	25	Page:	4
eq:BA	26	Page:	4
eq:xidef	27	Page:	4
eq:linsys	28	Page:	4
eq:kpq	29	Page:	4
eq:qpsi	30	Page:	4
eq:kpqeqe	31	Page:	4
eq:psi-main	32	Page:	4
eq:psi-mainc	33	Page:	4
eq:bilide	34	Page:	5
eq:biltau	35	Page:	5
eq:hirl	36	Page:	5
eq:hir-der	37	Page:	5
eq:hir2	38	Page:	5
eq:dress-tr	39	Page:	5
eq:hopf-co	40	Page:	5
eq:sato-lin	41	Page:	5
eq:hopf-coa	42	Page:	5
eq:wil	43	Page:	5
eq:hopf-cole	44	Page:	6
eq:lone	45	Page:	6
eq:wfaa	46	Page:	6
eq:faa	47	Page:	6
eq:mero	48	Page:	6
eq:panqi	49	Page:	6
eq:patwoqi	50	Page:	6
eq:papatwoqi	51	Page:	6
eq:lkplus	52	Page:	6
eq:pkplus	53	Page:	6
eq:partl	54	Page:	7
eq:L-3	55	Page:	7
eq:specA	56	Page:	7
eq:specb	57	Page:	7

2d	[1]
perel	[2]
hoppe	[3]
BV90	[4]
wadati	[5]
regge	[6]
Orlovetal	[7]
moerbeke	[8]
dickey	[9]
harada	[10]
ohta	[11]
chudnovsky	[12]
shiota	[13]
us	[14]

eq:lnn	58	Page:	7
eq:t1eqa	59	Page:	7
eq:t1eqb	60	Page:	7
eq:spectodc	61	Page:	7
eq:todadb	62	Page:	7
eq:lnoln1	63	Page:	7
eq:transf3	64	Page:	7
eq:specln	65	Page:	7